

The critical semilinear elliptic equation with isolated boundary singularities

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Abstract

We establish quantitative asymptotic behaviors for nonnegative solutions of the critical semilinear equation $-\Delta u = u^{\frac{n+2}{n-2}}$ with isolated boundary singularities, where $n \geq 3$ is the dimension.

1 Introduction

The internal isolated singularity for positive solutions of the semilinear equation $-\Delta u = u^p$ has been very well understood, where Δ is the Laplace operator, $1 < p \leq \frac{n+2}{n-2}$ is a parameter and $n \geq 3$ is the dimension. See Lions [21] for $1 < p < \frac{n}{n-2}$, Gidas-Spruck [13] for $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, Aviles [1] for $p = \frac{n}{n-2}$, Caffarelli-Gidas-Spruck [9] for $\frac{n}{n-2} \leq p \leq \frac{n+2}{n-2}$ and Korevaar-Mazzeo-Pacard-Schoen [16] for $p = \frac{n+2}{n-2}$. The Sobolev critical exponent $p = \frac{n+2}{n-2}$ case is of particular interest, because the equation connects to the Yamabe problem and the conformal invariance leads to a richer isolated singularity structure. See also Li [18] and Han-Li-Teixeira [14] for conformally invariant fully nonlinear elliptic equations.

The Dirichlet boundary isolated singularity for the same equation has also been studied in many cases. Asymptotic behaviors of singular solutions have been established by Bidaut-Véron-Vivier [5] for $1 < p < \frac{n+1}{n-1}$ and Bidaut-Véron-Ponce-Véron [3, 4] for $\frac{n+1}{n-1} \leq p < \frac{n+2}{n-2}$. Existence of singular solutions vanishing on boundaries of bounded domains except finite points has been obtained by del Pino-Musso-Pacard [12] for $p < \frac{n+2}{n-2}$. The exponent $\frac{n+1}{n-1}$ corresponding to $\frac{n}{n-2}$ for the interior singularity was discovered by Brézis-Turner [7]. Under a blow up rate assumption Bidaut-Véron-Ponce-Véron [3, 4] obtain refined asymptotic behaviors for the supercritical case $\frac{n+2}{n-2} < p < \frac{n+1}{n-3}$. We refer to [3] and references therein for related results on boundary singularity.

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This paper is concerned with the remaining critical case: $p = \frac{n+2}{n-2}$. The conformal invariance again produces additional complexity and the boundary condition makes the asymptotic analysis of [9] and [16] fail. As said in Bidaut-Véron-Ponce-Véron [4], one can show

Proposition 1.1. *Denote $\mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$. Let $u \in C^2(\mathbb{R}_+^n) \cap C(\bar{\mathbb{R}}_+^n \setminus \{0\})$ be a nonnegative solution of*

$$\begin{cases} -\Delta u = n(n-2)u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}_+^n, \\ u = 0 & \text{on } \partial\mathbb{R}_+^n \setminus \{0\}. \end{cases} \quad (1)$$

Suppose 0 is a non-removable singularity of u , then u depends only on $|x'|$ and x_n , and $\partial_r u(r, x_n) < 0$ for all $r = |x'| > 0$.

Note that nothing about the behavior of u at infinity is assumed in Proposition 1.1.

Let u be a solution of (1) and define $U(t, \theta) := |x|^{\frac{n-2}{2}} u(|x| \cdot \theta)$ with $t = -\ln |x|$. Then we have

$$\partial_{tt}^2 U + \Delta_{\mathbb{S}^{n-1}} U - \frac{(n-2)^2}{4} U + n(n-2)U^{\frac{n+2}{n-2}} = 0 \quad \text{on } \mathbb{R} \times \mathbb{S}_+^{n-1}, \quad (2)$$

$$U = 0 \quad \text{on } \mathbb{R} \times \partial\mathbb{S}_+^{n-1}, \quad (3)$$

where $\mathbb{S}_+^{n-1} = \{\theta = (\theta_1, \dots, \theta_n) \in \mathbb{S}^{n-1} : \theta_n > 0\}$. By Proposition 1.1, $U(t, \theta) = U(t, \theta_n)$. In contrast to the internal singularity studied by Caffarelli-Gidas-Spruck [9] and Korevaar-Mazzeo-Pacard-Schoen [16], we lose ODE analysis to classify all solutions of equation (2)-(3). del Pino-Musso-Pacard [12] conjectured that there exists a one-parameter family of solutions of (2)-(3). Bidaut-Véron-Ponce-Véron [3, 4] proved that there exists a unique t -independent solution. Existence of t -dependent solutions and a priori estimates are left open.

Let ψ be a C^2 function in \mathbb{R}^{n-1} satisfying

$$\psi(0) = 0, \quad \nabla \psi(0) = 0.$$

Let $Q_R = \{x = (x', x_n) : x_n > \psi(x')\} \cap B_R$ and $\Gamma_R = \{x = (x', x_n) : x_n = \psi(x')\} \cap B_R$, where B_R is the open ball center at 0 with radius R . We consider nonnegative solutions of

$$\begin{cases} -\Delta u = n(n-2)u^{\frac{n+2}{n-2}} & \text{in } Q_1, \\ u = 0 & \text{on } \Gamma_1 \setminus \{0\}. \end{cases} \quad (4)$$

Theorem 1.2. *Let $u \in C^2(\bar{Q}_1 \setminus \{0\})$ be a nonnegative solution of (4). Then for each $0 < \gamma < 1$ there exists a constant $C(\gamma) \geq 1$ such that for all $x \in Q_{1/2}$ with $\frac{d(x)}{|x|} \leq \gamma$,*

$$u(x) \leq C(\gamma)d(x)|x|^{-\frac{n}{2}} \quad (5)$$

and

$$u(x', x_n) = \bar{u}(|x'|, x_n)(1 + O(|x|)) \quad \text{if } x_n > \frac{1}{\delta} \max_{|y'|=|x'|} |\psi(y')|, \quad (6)$$

where $d(x) = \text{dist}(x, \Gamma_1)$, $\bar{u}(x', x_n) = \int_{\mathbb{S}^{n-2}} u(|x'|\theta, x_n) d\theta$, $O(|x|) \leq C(\gamma)|x|$ as $x \rightarrow 0$, and $\delta > 0$ depends only on the C^2 norm of ψ .

If the above inequality (5) holds for $\gamma = 1$, then either 0 is a removable singularity or there exists a constant $C > 0$ such that

$$u(x) \geq \frac{1}{C} d(x) |x|^{-\frac{n}{2}} \quad \text{for all } x \in Q_{1/2}. \quad (7)$$

Furthermore, (6) holds for all $x \in Q_{1/2}$.

Without assuming (5) holds up to $\gamma = 1$, we are still able to show some sort of asymptotic symmetry for almost all $x \in Q_{1/2}$ close to 0; see Proposition 5.1.

The second conclusion of Theorem 1.2 is a partial answer of a question of [4] (see Remark 1 of [4]).

The method of proof of Theorem 1.2 can be adapted to study boundary singularity of critical equations with nonlinear Neumann boundary conditions, which we leave to another paper. Motivated by conformal geometry, boundary singularity of linear (degenerate) elliptic equation with a critical nonlinear Neumann boundary condition has been studied in Caffarelli-Jin-Sire-Xiong [10], Jin-de Queiroz-Sire-Xiong [15], and Sun-Xiong [23].

The organization of paper and crucial steps of the proofs are as follows. In section 2, we recall some basic facts of elliptic equations with zero Dirichlet condition, such as boundary Harnack inequality, a special maximum principle and a boundary version of Bôcher theorem. In section 3, we prove the partial upper bound in the main theorem. A classical result of Berestycki-Nirenberg [2] plays an important role. In section 4, we establish the lower bound. The Pohozaev identity and Bôcher theorem are used crucially. In section 5, we prove symmetry results via the moving spheres method developed by Li-Zhu [20]; see also Li-Zhang [19]. In particular, we present a proof of Proposition 1.1 by this method. Here the boundary Harnack inequality is used repeatedly. Our proof of Proposition 1.1 can be adapted to give another proof of a result of Dancer [11].

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2 Preliminaries

Since we will work in neighborhoods of partial boundaries of domains, we let ψ be a C^2 function in \mathbb{R}^{n-1} satisfying

$$\psi(0) = 0, \quad \nabla \psi(0) = 0$$

and denote $Q_R = \{x = (x', x_n) : x_n > \psi(x')\} \cap B_R$, $\Gamma_R = \{x = (x', x_n) : x_n = \psi(x')\} \cap B_R$ and $d(x) = \text{dist}(x, \Gamma_1)$, where B_R is the open ball center at 0 with radius R . We denote $B_R^+(y)$ for $y = (y', 0)$ as $\{x \in B_R(y) : x_n > 0\}$, $\partial' B_R^+(y) := B_R(y) \cap \partial \mathbb{R}_+^n$ and $\partial'' B_R^+(y) := B_R(y) \cap \mathbb{R}_+^n$. We assume

$$\|\psi\|_{C^2(\partial' B_1^+)} \leq A_1.$$

Lemma 2.1. *Let $u \in C^2(\bar{Q}_1)$ be a nonnegative solution of*

$$\begin{cases} -\Delta u = a(x)u & \text{in } Q_1, \\ u = 0 & \text{on } \Gamma_1, \end{cases}$$

where $a(x) \in L^\infty(Q_1)$. Then

$$\frac{u(x)}{d(x)} \leq C_0 \frac{u(y)}{d(y)} \quad \text{for } x, y \in Q_{1/2}$$

and

$$\left| \frac{u(x)}{d(x)} - \frac{u(y)}{d(y)} \right| \leq C_0 |x - y|^\alpha \quad \text{for } x, y \in Q_{1/2},$$

where $C_0 \geq 1$ and $\alpha \in (0, 1)$ are constants depending only on n , A_1 , $\|a\|_{L^\infty(Q_1)}$ and $\|u\|_{L^\infty(Q_1)}$.

The above result is a simple version of the boundary Harnack inequality; see Caffarelli-Fabes-Mortola-Salsa [8] or Krylov [17]. If only a differential inequality is assumed, we have

Lemma 2.2. *Let $u \in C^2(\bar{Q}_1 \setminus \{0\})$ be a nonnegative solution of*

$$\begin{cases} -\Delta u \geq 0 & \text{in } Q_1, \\ u = 0 & \text{on } \Gamma_1 \setminus \{0\}, \end{cases}$$

If $u > 0$ somewhere in Q_1 , then there exists a constant $c_0 > 1$ such that

$$u(x) \geq d(x)/c_0 \quad \text{for all } x \in Q_{1/2}.$$

Proof. It is easy to check that there exist constants $0 < \delta < 1/4$ and $A \geq 1$ with $A\delta < 1/2$ such that

$$\Delta\left(\frac{d(x)}{1 - Ad(x)}\right) \geq 0 \quad \text{for } x \in Q_{1/2}, \quad d(x) < \delta. \quad (8)$$

It follows from the strong maximum principle that $u > 0$ in Q_1 . By Hopf Lemma and the compactness of $\Gamma_1 \cap \partial B_{1/2}$, there exists a constant $b_0 > 0$ such that

$$-\partial_\nu u \geq b_0 > 0 \quad \text{on } \Gamma_1 \cap \partial B_{1/2}.$$

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By the continuity of ∇u on $Q_1 \cap \partial B_{1/2}$, one can find a constant $C > 0$ such that

$$u(x) \geq \frac{1}{C} \frac{d(x)}{1 - Ad(x)} \quad \text{on } \partial(Q_{1/2} \cap \{d(x) < \delta\}).$$

In view of (8), by maximum principle we have

$$u(x) \geq \frac{1}{C} \frac{d(x)}{1 - Ad(x)} \quad \text{in } Q_{1/2} \cap \{d(x) < \delta\}.$$

The lemma follows immediately. □

We will also use a maximum principle when the coefficient of zero order term without sign restriction but being small.

Lemma 2.3. *Let $u \in C^2(\bar{Q}_1 \setminus \{0\})$ be a solution of*

$$\Delta u + au \leq 0 \quad \text{in } Q_1, \quad u = 0 \quad \text{on } \Gamma_1 \setminus \{0\}.$$

There exists a small constant $\delta = \delta(n) > 0$ such that if $|a(x)| \leq \delta|x|^{-2}$, and $u \geq 0$ on $\partial(Q_1 \setminus Q_\varepsilon)$ for some $\varepsilon > 0$, there holds

$$u \geq 0 \quad \text{in } Q_1 \setminus Q_\varepsilon.$$

Proof. Multiplying both sides by $u^- = \max\{-u, 0\}$ and using the Hardy inequality, the lemma follows immediately. □

Lemma 2.4. *For $R > 0$, let $u \in C^2(\mathbb{R}_+^n \setminus B_R)$ and be continuous up to the boundary $\partial\mathbb{R}_+^n \setminus B_R$ except a point $\bar{x} = (\bar{x}', 0)$ with $|\bar{x}'| > R$. Suppose that*

$$-\Delta u \geq 0 \quad \text{in } \mathbb{R}_+^n \setminus B_R^+, \quad u(x', 0) = 0 \quad \text{for } x' \neq \bar{x}',$$

and $u \geq 0$. Then

$$u(x) \geq \frac{R^n x_n}{|x|^n} \inf_{y \in \partial'' B_R^+} \frac{u(y)}{y_n} \quad \text{in } \mathbb{R}_+^n \setminus B_R^+.$$

Proof. For $\varepsilon > 0$, let

$$\phi_\varepsilon(x) = \frac{R^n x_n}{|x|^n} \inf_{y \in \partial'' B_R^+} \frac{u(y)}{y_n} - \varepsilon.$$

By maximum principle we have $u \geq \phi_\varepsilon$. Sending $\varepsilon \rightarrow 0$, the lemma follows. □

To establish the lower bound in Theorem 1.2, we need a well-known boundary Bôcher type theorem. See Marcus-Véron [22] for a nonlinear version.

Lemma 2.5. *Let $u \in C^2(B_1^+) \cap C^0(\bar{B}_1^+ \setminus \{0\})$ be a nonnegative solution of*

$$-\Delta u = 0 \quad \text{in } B_1^+, \quad u(x', 0) = 0 \quad \text{for } x' \neq 0.$$

Then

$$u(x) = a \frac{x_n}{|x|^n} + h(x),$$

where $a \geq 0$ is a constant and

$$-\Delta h = 0 \quad \text{in } B_1^+, \quad h(x', 0) = 0 \quad \text{for } |x'| < 1.$$

Furthermore, if B_1^+ is replaced by \mathbb{R}_+^n , then $h = bx_n$ for some nonnegative constant b .

3 A partial upper bound

The following lemma is an easy consequence of a classical result of Berestycki-Nirenberg [2].

Lemma 3.1. *Let Ω be a bounded domain in \mathbb{R}^n , which is convex in the x_1 direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Let $u \in C^2(\Omega)$ be a positive solution of*

$$-\Delta u = f(u) \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega \cap \{x_1 > 0\},$$

where $f : [0, \infty) \rightarrow [0, \infty)$ is local Lipschitz continuous. Then $\partial_{x_1} u(x) < 0$ for $x \in \Omega$ with $x_1 > 0$.

In order to use Lemma 3.1, let us define the conformal transform $F : \mathbb{R}_+^n \rightarrow B_1$ by

$$F(x) := \left(\frac{2x'}{|x'|^2 + (1 + x_n)^2}, \frac{|x|^2 - 1}{|x'|^2 + (1 + x_n)^2} \right).$$

Then $F|_{\partial\mathbb{R}_+^n}$ is the inverse of stereographic projection. Let u be a solution of (4), then the function

$$v(z) := |J_F|^{-\frac{n-2}{2n}} u(x)$$

satisfies

$$-\Delta v = v^{\frac{n+2}{n-2}} \quad \text{in } F(Q_1 \cap B_1^+),$$

where $z := F(x)$ and $|J_F|$ is the Jacobian determinant of F .

By performing a Kelvin transform with respect to a sphere of small radius below Q_1 and re-labeling coordinates, we may assume that ψ is convex. Since $\psi(0) = 0$ and $\nabla\psi(0) = 0$, we have $\psi \geq 0$. For $0 < r < 1$, denote

$$\tilde{Q}_{1/r} = F\left(\frac{1}{r}Q_1\right), \quad \tilde{\Gamma}_{1/r} = F\left(\frac{1}{r}\Gamma_1\right)$$

and

$$v(F(x)) = |J_F|^{-\frac{n-2}{2n}} r^{\frac{n-2}{2}} u(rx).$$

Then

$$-\Delta v = n(n-2)v^{\frac{n+2}{n-2}} \quad \text{in } \tilde{Q}_{1/r}, \quad v = 0 \quad \text{on } \tilde{\Gamma}_{1/r} \setminus \{-e_n\}, \quad (9)$$

where $e_n = (0', 1)$. Note that $rx \in Q_1$ implies $x \in \{y_n > \frac{1}{r}\psi(r'y')\}$. Clearly,

$$\begin{aligned} (z', z_n) &:= \left(\frac{2x'}{|x'|^2 + (1 + r^{-1}\psi(rx'))^2}, \frac{|x'|^2 + r^{-2}\psi(rx')^2 - 1}{|x'|^2 + (1 + r^{-1}\psi(rx'))^2} \right) \\ &\rightarrow \left(\frac{2x'}{|x'|^2 + 1}, \frac{|x'|^2 - 1}{|x'|^2 + 1} \right) \quad \text{in } C_{loc}^2(\mathbb{R}^{n-1}) \text{ as } r \rightarrow 0 \end{aligned} \quad (10)$$

and thus

$$\tilde{\Gamma}_{1/r} \rightarrow \partial B_1 \setminus \{e_n\}. \quad (11)$$

Now we able to prove a partial upper bound.

Proposition 3.2. *Let $u \in C^2(\bar{Q}_1 \setminus \{0\})$ be a nonnegative solution of (4). Then for every $0 < \gamma < 1$, there exists a constant $C(\gamma) > 0$ such that*

$$|x|^{\frac{n-2}{2}} u(x) \leq C(\gamma) \quad \forall x \in Q_{3/4} \text{ with } \frac{d(x)}{|x|} < \gamma.$$

Proof. By (10) and (11), for any constant $0 < \delta < 1/2$ there exists a constant $r_0 > 0$ such that $\tilde{Q}_{1/r} \cap \{z_n < 1 - \delta\}$ is a convex body for all $0 < r < r_0$. Together with Lemma 3.1, there exists a constant $c(\delta) > \delta$ with $\lim_{\delta \rightarrow 0} c(\delta) = 0$, depending only on n and δ , such that v has no critical points in the region $\tilde{Q}_{1/r} \setminus \mathcal{C}_{c(\delta)}$, where $\mathcal{C}_{c(\delta)}$ is the cone generated by the vertex $-e_n$ and $B_{c(\delta)}(e_n)$. Choose δ small such that

$$F(\{x \in Q_{1/4} : \frac{x_n}{|x|} \leq \gamma\}) \cap \mathcal{C}_{2c(\delta)} = \emptyset.$$

Since F is a conformal map,

$$\text{dist}(F(x), -e_n) = 2|x| \cdot \frac{\sqrt{|x|^2 + 2x_n + 1}}{|x'|^2 + (1 + x_n)^2} \approx |x| \quad (12)$$

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and $|J_F(x)|$ is smooth positive smooth function on $\bar{B}_{1/2}^+$, we only need show that for $0 < r < r_0$

$$\limsup_{\tilde{Q}_{1/r} \setminus \mathcal{C}_{2c(\delta)} \ni z \rightarrow -e_n} |z|^{\frac{n-2}{2}} v(z) < \infty. \quad (13)$$

Suppose the contrary that there exists a sequence $\{z_j\}_{j=1}^\infty \subset \tilde{Q}_{1/r} \setminus \mathcal{C}_{2c(\delta)}$ such that

$$z_j \rightarrow -e_n \quad \text{as } j \rightarrow \infty,$$

and

$$|z_j|^{\frac{n-2}{2}} v(z_j) \rightarrow \infty \quad \text{as } j \rightarrow \infty. \quad (14)$$

Let $\theta = \arcsin c(\delta) - \arcsin \frac{c(\delta)}{2} > 0$ be the cone angle error between $\mathcal{C}_{2c(\delta)}$ and $\mathcal{C}_{c(\delta)}$. It is easy to see that

$$B_{|z_j| \sin \theta}(z_j) \cap \mathcal{C}_{c(\delta)} = \emptyset. \quad (15)$$

Consider

$$v_j(z) := \left(\frac{\sin \theta}{2} |z_j| - |z - z_j| \right)^{\frac{n-2}{2}} v(z), \quad |z - z_j| \leq \frac{\sin \theta}{2} |z_j|.$$

Let $|\bar{z}_j - z_j| < \frac{\sin \theta}{2} |z_j|$ satisfy

$$v_j(\bar{z}_j) = \max_{|z - z_j| \leq \frac{\sin \theta}{2} |z_j|} v_j(z),$$

and let

$$2\mu_j := \frac{\sin \theta}{2} |z_j| - |\bar{z}_j - z_j|.$$

Then

$$0 < 2\mu_j \leq \frac{\sin \theta}{2} |z_j| \quad \text{and} \quad \frac{\sin \theta}{2} |z_j| - |z - z_j| \geq \mu_j \quad \forall |z - \bar{z}_j| \leq \mu_j. \quad (16)$$

By the definition of v_j , we have

$$(2\mu_j)^{\frac{n-2}{2}} v(\bar{z}_j) = v_j(\bar{z}_j) \geq v_j(z) \geq (\mu_j)^{\frac{n-2}{2}} v(z) \quad \forall |z - \bar{z}_j| \leq \mu_j. \quad (17)$$

Thus, we have

$$2^{\frac{n-2}{2}} v(\bar{z}_j) \geq v(z) \quad \forall |z - \bar{z}_j| \leq \mu_j.$$

We also have

$$(2\mu_j)^{\frac{n-2}{2}} v(\bar{z}_j) = v_j(\bar{z}_j) \geq v(z_j) = \left(\frac{\sin \theta |z_j|}{2} \right)^{\frac{n-2}{2}} v(z_j) \rightarrow \infty \quad \text{as } j \rightarrow \infty. \quad (18)$$

Now, consider

$$w_j(y) = \frac{1}{v(\bar{z}_j)} v(\bar{z}_j + \frac{y}{v(\bar{z}_j)^{\frac{n-2}{2}}}), \quad y \in \Omega_j,$$

where

$$\Omega_j := \left\{ y \in \mathbb{R}^n \mid \bar{z}_j + \frac{y}{v(\bar{z}_j)^{\frac{2}{n-2}}} \in \tilde{Q}_{1/r} \right\}.$$

Then w_j satisfies $w(0) = 1$ and

$$-\Delta w_j = w_j^{\frac{n+2}{n-2}} \quad \text{in } \Omega_j \quad (19)$$

Moreover, it follows from (17) and (18) that

$$w_j(y) \leq 2^{\frac{n-2}{2}} \quad \text{in } B_{R_j} \cap \Omega_j,$$

where

$$R_j := \mu_j v(\bar{z}_j)^{\frac{2}{n-2}} \rightarrow \infty \text{ as } j \rightarrow \infty.$$

Since ψ is C^2 and $\nabla \psi(0) = 0$, we have two possibilities:

If $B_{R_j} \cap \Omega_j \rightarrow \mathbb{R}^n \cap \{x_n > -\rho\}$ for some $\rho > 0$, then by the up to boundary estimates for 2nd order linear elliptic equations there exists a subsequence of $\{w_j\}$, which is still denoted as $\{w_j\}$, satisfying

$$w_j \rightarrow w \quad \text{in } C_{loc}^2(\mathbb{R}^n \cap \{y_n \geq \rho\})$$

for some w satisfying

$$-\Delta w = n(n-2)w^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n \cap \{y_n \geq \rho\}, \quad w(y', \rho) = 0. \quad (20)$$

By [11], $w = 0$. This is impossible since $w(0) = 1$.

If $B_{R_j} \cap \Omega_j \rightarrow \mathbb{R}^n$, then by the interior estimates for 2nd order linear elliptic equations there exists a subsequence of $\{w_j\}$, which is still denoted as $\{w_j\}$, satisfying

$$w_j \rightarrow w \quad \text{in } C_{loc}^2(\mathbb{R}^n)$$

for some w satisfying

$$-\Delta w = n(n-2)w^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n. \quad (21)$$

By [9] we have

$$w(y) = \left(\frac{\lambda}{1 + \lambda^2 |y - \bar{y}|^2} \right)^{\frac{n-2}{2}} \quad (22)$$

for some point $\bar{y} \in \mathbb{R}^n$ and $1 \leq \lambda \leq 2^{\frac{n-2}{2}}$.

Since $w_j \rightarrow w$ in $C_{loc}^2(\mathbb{R}^n)$ and $\nabla w(\bar{y}) = 0$ and $\nabla^2 w(\bar{x})$ is negative definite, for large j there exists $y_j \in B_1(\bar{y})$ such that $\nabla w_j(y_j) = 0$. By the definition of w_j , we have

$$\nabla v(\bar{z}_j + \frac{y_j}{u(\bar{z}_j)^{\frac{2}{n-2}}}) = 0.$$

This is impossible, since $\bar{z}_j + \frac{y_j}{u(\bar{z}_j)^{\frac{n-2}{2}}} \in \tilde{Q}_{1/r} \setminus \mathcal{C}_{c(\delta)}$ where v does not have any critical point. Therefore, (13) holds. Scaling back to u and using (12), we proved Proposition 3.2. \square

Corollary 3.3. *Assume the assumptions in Proposition 3.2. Let $0 < \gamma < 1$. Then for $x \in Q_{1/4}$ with $\frac{d(x)}{|x|} \leq \gamma$,*

$$u(x) \leq C(\gamma)d(x)|x|^{-\frac{n}{2}}, \quad (23)$$

$$|\nabla^k u| \leq C(\gamma)|x|^{-\frac{n-2}{2}-k}, \quad k = 0, 1, 2, \quad (24)$$

and for any $0 < s < \frac{1}{4}$, $x, y \in Q_{2s} \setminus Q_{s/2}$ with $\frac{d(x)}{|x|}, \frac{d(y)}{|y|} \leq \gamma$,

$$\frac{u(x)}{d(x)} \leq C(\gamma)\frac{u(y)}{d(y)}, \quad (25)$$

where $r = |x|$, $\theta = \frac{x}{|x|}$, $C(\gamma) > 0$ depends on γ but not s .

Proof. For $0 < s < \frac{1}{4}$, let

$$v_s(x) = s^{\frac{n-2}{2}} u(sx).$$

Denote $\tilde{Q}_r := \{x : sx \in Q_{rs}\}$, $\tilde{\Gamma}_r := \{x : sx \in \Gamma_{rs}\}$ for any $r > 0$, and $\tilde{d}(x) = \text{dist}(x, \tilde{\Gamma}_1)$. Then

$$-\Delta v_s = n(n-2)v_s^{\frac{n+2}{n-2}} \quad \text{in } \tilde{Q}_3 \setminus \tilde{Q}_{1/4}, \quad v_s = 0 \quad \text{on } \Gamma_3 \setminus \Gamma_{1/4}.$$

It follows from Proposition 3.2 that $v_s(x) \leq C(\gamma')$ for $x \in \tilde{Q}_3 \setminus \tilde{Q}_{1/4}$ with $\frac{\tilde{d}(x)}{|x|} < \gamma'$, where $0 < \gamma < \gamma' < 1$. By Lemma 2.1 and the standard linear elliptic equations theory, for $x, y \in \tilde{Q}_2 \setminus Q_{1/2}$ with $\frac{\tilde{d}(x)}{|x|} \leq \gamma$ and $\frac{\tilde{d}(y)}{|y|} \leq \gamma$ we have

$$v_s(x) \leq \tilde{d}(x),$$

$$|\nabla^k v_s(x)| \leq C, \quad k = 1, 2,$$

$$\frac{v_s(x)}{\tilde{d}(x)} \leq C \frac{v_s(y)}{\tilde{d}(y)},$$

where C depends only on $n, C(\gamma')$. Scaling back to u , the above three inequalities yield (23), (24) and (25), respectively.

Therefore, we complete the proof. \square

Remark 3.4. *If $u(x) \leq C|x|^{\frac{2-n}{2}}$ for all $x \in Q_{1/4}$, Corollary 3.3 holds for $\gamma = 1$.*

4 A lower bound and removability

Lemma 4.1 (Pohozaev identity). *Let $u \in C^2(\bar{Q}_1 \setminus \{0\})$ be a nonnegative solution of (4). Then for all $0 < r < 1$ there holds*

$$P(u, r) := \int_{Q_1 \cap \partial B_r} \frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2} |\nabla u|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 + \frac{(n-2)^2}{8} r u^{\frac{2n}{n-2}} dS = c_0,$$

where c_0 is constant independent of r .

The proof of Lemma 4.1 is standard by now. $P(u, r)$ is called *Pohozaev integral* sometimes in the literature.

Proposition 4.2. *Let $u \in C^2(\bar{Q}_1 \setminus \{0\})$ be a nonnegative solution of (4). If*

$$\limsup_{Q_1 \ni x \rightarrow 0} |x|^{\frac{n-2}{2}} u(x) < \infty \quad (26)$$

and

$$\liminf_{Q_1 \ni x \rightarrow 0} d(x)^{-1} |x|^{\frac{n}{2}} u(x) = 0,$$

then

$$\lim_{Q_1 \ni x \rightarrow 0} d(x)^{-1} |x|^{\frac{n}{2}} u(x) = 0.$$

Proof. Suppose the contrary that

$$\limsup_{Q_1 \ni x \rightarrow 0} d(x)^{-1} |x|^{\frac{n}{2}} u(x) = C_0 > 0.$$

Since $\liminf_{Q_1 \ni x \rightarrow 0} d(x)^{-1} |x|^{\frac{n}{2}} u(x) = 0$, by the Harnack inequality (25) in annulus we can find sequences $x_j = (0', (x_j)_n) \rightarrow 0$ and $y_j = (0', (y_j)_n) \rightarrow 0$ as $j \rightarrow \infty$ satisfying

$$|x_j|^{\frac{n-2}{2}} u(x_j) \rightarrow 0 \quad \text{and} \quad |y_j|^{\frac{n-2}{2}} u(y_j) \rightarrow C_0^* \quad \text{as } j \rightarrow \infty,$$

where $0 < \bar{C}_0^* \leq C_0$. In view of this oscillation picture, without loss of generality we assume $(x_j)_n$ are local minimum of $x_n^{-1} |x|^{\frac{n}{2}} u(x)$ restricted to the line $(0', x_n)$. It follows that

$$\frac{\partial}{\partial r} (x_n^{-1} |x|^{\frac{n}{2}} u(x)) \Big|_{x=x_j} = 0. \quad (27)$$

Let $r_j = |x_j| = (x_j)_n > 0$, and

$$w_j(x) = \frac{u(r_j x)}{u(x_j)}.$$

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Denote $\tilde{Q}_j := \{x : r_j x \in Q_1\}$, $\tilde{\Gamma}_j := \{x : r_j x \in \Gamma_1\}$, and $\tilde{d}_j(x) = \text{dist}(x, \tilde{\Gamma}_j)$. It follows from (25) that for any $R > 1$ there exists $C(R) > 0$ such that

$$w_j(x) \leq C(R)\tilde{d}_j(x) \quad \forall \frac{1}{R} \leq |x| \leq R \text{ and large } j.$$

Furthermore, w_j satisfies

$$-\Delta w_j = n(n-2)(r_j^{\frac{n-2}{2}} u(x_j))^{\frac{4}{n-2}} w_j^{\frac{n+2}{n-2}} \quad \text{in } \tilde{Q}_j,$$

$$w_j = 0 \quad \text{on } \tilde{\Gamma}_j \setminus \{0\}.$$

By the up to boundary estimates for linear elliptic equation, after passing to a subsequence, as $j \rightarrow \infty$,

$$w_j \rightarrow w \quad \text{in } C_{loc}^2(\bar{R}_+^n \setminus \{0\}),$$

and $0 < w \in C_{loc}^2(\bar{R}_+^n \setminus \{0\})$ satisfies

$$-\Delta w = 0 \quad \text{in } \mathbb{R}_+^n,$$

$$w(x', 0) = 0 \quad \text{for all } x' \neq 0.$$

By Lemma 2.5,

$$w(x) = a \frac{x_n}{|x|^n} + b x_n,$$

where $a, b \geq 0$ are constants. By (27) and $w_j(e_n) = 1$, we have

$$0 = \frac{\partial}{\partial r}(x_n^{-1}|x|^{\frac{n}{2}} w(x)) \Big|_{x=e_n} = \frac{n}{2}(b-a)$$

and $a + b = 1$. Thus $a = b = \frac{1}{2}$.

By (23) and (24), we have

$$\lim_{r_j \rightarrow 0} P(u, r_j) = 0.$$

It follows from Lemma 4.1 that

$$P(u, r_j) = 0 \quad \text{for all } j.$$

On the other hand,

$$0 = P(u, r_j) = P(r_j^{\frac{n-2}{2}} u(r_j x), 1) = P(r_j^{\frac{n-2}{2}} u(x_j) w_j(x), 1).$$

Therefore, as $j \rightarrow \infty$

$$\begin{aligned}
 0 &= \int_{\bar{Q}_j \cap \partial B_1^+} \frac{n-2}{2} w_j \frac{\partial w_j}{\partial r} - \frac{1}{2} |\nabla w_j|^2 + \left| \frac{\partial w_j}{\partial r} \right|^2 + \frac{(n-2)^2}{8} (r_j^{\frac{n-2}{2}} u(x_j))^{\frac{4}{n-2}} w_j^{\frac{2n}{n-2}} dS \\
 &\rightarrow \int_{\partial'' B_1^+} \frac{n-2}{2} w \frac{\partial w}{\partial r} - \frac{1}{2} |\nabla w|^2 + \left| \frac{\partial w}{\partial r} \right|^2 dS \\
 &= -\frac{1}{4n} |\mathbb{S}^{n-1}|.
 \end{aligned}$$

We obtain a contradiction. The proposition is proved. \square

Proposition 4.3. *Let $u \in C^2(\bar{Q}_1 \setminus \{0\})$ be a nonnegative solution of (4). If*

$$\lim_{Q_1 \ni x \rightarrow 0} d(x)^{-1} |x|^{\frac{n}{2}} u(x) = 0, \quad (28)$$

then 0 is a removable singular point of u .

Proposition 4.3 is included in Theorem 7.1 of [3]. We provide

Another proof of Proposition 4.3. Since the critical equation is conformally invariant, we may assume Q_σ is convex for some small $\sigma > 0$, otherwise one may perform a kelvin transform centered in Q_1 . For any $0 < \mu \leq n$, we have

$$\Delta \frac{x_n}{|x|^\mu} = -\mu(n-\mu) |x|^{-(\mu+2)} x_n.$$

For any $\varepsilon > 0$, let

$$\phi_\varepsilon = \alpha \frac{x_n}{|x|} + \varepsilon \frac{x_n}{|x|^{n-1}},$$

where $\alpha > 0$ is a constant to be fixed. Hence,

$$(\Delta + (n-1)|x|^{-2})\phi_\varepsilon = 0 \quad \text{in } B_1^+, \quad \phi_\varepsilon(x', 0) = 0 \quad \text{for } x' \neq 0.$$

By (28), for any $\delta > 0$ there exists $\tau > 0$ such that

$$a(x) := n(n-2)u(x)^{\frac{4}{n-2}} \leq \delta |x|^{-2} \quad \text{for } |x| \leq \tau < \sigma.$$

Therefore, we have

$$(\Delta + a(x))(\phi_\varepsilon - u)(x) = -((n-1)|x|^{-2} - a(x))\phi_\varepsilon(x) < 0 \quad \text{in } Q_\tau.$$

Choose $\delta < n - 1$ small and thus the assumptions in Lemma 2.3 are satisfied. Thanks to Lemma 2.1, one can choose α such that $\alpha \frac{x_n}{|x|} \geq u$ on $\partial B_\tau^+ \cap Q_\tau$. Since Q_τ is convex, $\phi_\varepsilon \geq 0 = u$ on the bottom boundary of Q_τ . By (28), we have

$$\liminf_{x \rightarrow 0} (\phi_\varepsilon - u)(x) \geq 0.$$

It follows from Lemma 2.3 that $u \leq \phi_\varepsilon$ in Q_τ for all $\varepsilon > 0$. Sending $\varepsilon \rightarrow 0$, we have

$$u \leq \alpha \frac{x_n}{|x|} \quad \forall x \in Q_\tau.$$

It follows that 0 is a removable singularity. We complete the proof. □

5 Asymptotic symmetry and proof of Theorem 1.2

Proposition 5.1. *Let $u \in C^2(\bar{Q}_1 \setminus \{0\})$ be a nonnegative solution of (4). Suppose that ψ is concave and 0 is a non-removable singularity. Then there exists $\varepsilon > 0$ such that for every $x = (x', x_n) \in \Gamma_1$ with $|x'| < \varepsilon$ there holds*

$$u_{x,\lambda}(y) \leq u(y) \quad \forall 0 < \lambda < |x'|, y \in Q_{3/4} \setminus B_\lambda^+(x),$$

where

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y - x|} \right)^{n-2} u\left(x + \frac{\lambda^2(y - x)}{|y - x|^2}\right).$$

Proof. To present our idea more clear, let us assume $\psi = 0$ at the moment. The proposition is proved as long as the three steps have been through:

- (a). There exists $0 < \varepsilon < 1/10$ such that for every $x = (x', 0)$ with $|x'| < \varepsilon$

$$u_{x,\lambda}(y) < u(y) \quad \forall 0 < \lambda < |x'|, y \in \partial'' B_{3/4}^+.$$

- (b). There exists $0 < \lambda_1 < |x|$ such that

$$u_{x,\lambda}(y) \leq u(y) \quad \forall 0 < \lambda < \lambda_1, y \in B_{3/4}^+ \setminus B_\lambda^+(x).$$

- (c). Let

$$\bar{\lambda}_x = \sup\{0 < \lambda < |x| : u_{x,\mu}(y) \leq u(y) \quad \forall 0 < \mu < \lambda, y \in B_{3/4}^+ \setminus B_\mu^+(x)\}.$$

Then $\bar{\lambda}_x = |x|$.

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Proof of (a). By Lemma 2.2, there exists a constant $c_0 > 0$ such that

$$u(y) \geq c_0 y_n \quad \text{on } B_{3/4}^+. \quad (29)$$

For $0 < \lambda < |x| < \varepsilon$, $y \in \partial'' B_{3/4}$, we have

$$\left| x + \frac{\lambda^2(y-x)}{|y-x|^2} \right| \geq |x| - \frac{20}{13}|x|^2 \geq \frac{11}{13}|x|,$$

and

$$\frac{\lambda^2 y_n}{|y-x|^2} \leq \frac{20}{13}|x|^2 y_n.$$

Thus

$$\frac{(x + \frac{\lambda^2(y-x)}{|y-x|^2})_n}{|x + \frac{\lambda^2(y-x)}{|y-x|^2}|} \leq \frac{15}{11}|x| < \frac{3}{22}.$$

It follows from Corollary 3.3 with $\gamma = \frac{3}{22}$ that

$$u(x + \frac{\lambda^2(y-x)}{|y-x|^2}) \leq C y_n |x|^{-\frac{n-4}{2}}.$$

Hence

$$u_{x,\lambda}(y) = \left(\frac{\lambda}{|y-x|} \right)^{n-2} u(x + \frac{\lambda^2(y-x)}{|y-x|^2}) \leq C y_n \lambda^{n-2} |x|^{-\frac{n-4}{2}} \leq C y_n |x|^{\frac{n}{2}} < c_0 y_n,$$

provided

$$0 < \varepsilon < \left(\frac{c_0}{C} \right)^{\frac{2}{n}}.$$

Proof of (b). For any fixed $0 < |x| < \varepsilon$, we claim there exist $0 < \lambda_3 < \lambda_2 < |x|$ such that

$$u_{x,\lambda}(y) \leq u(y) \quad \forall 0 < \lambda \leq \lambda_3, y \in B_{3/4}^+(x) \setminus B_{\lambda_2}^+(x). \quad (30)$$

Indeed, for every $0 < \lambda_2 < |x|$ and every $y \in B_{3/4}^+ \setminus B_{\lambda_2}^+(x)$, we have $x + \frac{\lambda^2(y-x)}{|y-x|^2} \in B_{\lambda_2}^+(x)$. Hence,

$$\begin{aligned} u_{x,\lambda}(y) &= \left(\frac{\lambda}{|y-x|} \right)^{n-2} u(x + \frac{\lambda^2(y-x)}{|y-x|^2}) \\ &\leq \left(\frac{\lambda}{|y-x|} \right)^{n-2} \frac{\lambda^2 y_n}{|y-x|^2} \sup_{z \in B_{\lambda_2}^+(x)} \frac{u(z)}{z_n} \\ &\leq c_0 y_n \leq u(y) \end{aligned}$$

where c_0 is the constant in (29),

$$0 < \lambda_3 = \lambda_2 \left(c_0 / \sup_{z \in B_{\lambda_2}^+(x)} \frac{u(z)}{z_n} \right)^{1/n},$$

and Lemma 2.1 has been used. Therefore, (30) is confirmed.

We are going to use the *narrow domain technique* to conclude that the remaining case: $u_{x,\lambda} \leq u$ in $B_{\lambda_2}^+(x) \setminus B_{\lambda}^+(x)$.

By (30), $u_{x,\lambda} \leq u$ on $\partial(B_{\lambda_2}^+(x) \setminus B_{\lambda}^+(x))$ for all $0 < \lambda < \lambda_3$. Multiplying both sides of the equation

$$-\Delta(u_{x,\lambda} - u) = n(n-2)(u_{x,\lambda} + \tau u)^{\frac{4}{n-2}}(u_{x,\lambda} - u) \quad \text{in } B_{\lambda_2}^+(x) \setminus B_{\lambda}^+(x)$$

by $(u_{x,\lambda} - u)^+$ and integrating by parts, where we used mean value theorem and $0 \leq \tau = \tau(x) \leq 1$, we have, using Hölder inequality,

$$\begin{aligned} & \int_{B_{\lambda_2}^+(x) \setminus B_{\lambda}^+(x)} |\nabla(u_{x,\lambda} - u)^+|^2 \\ &= n(n-2) \int_{B_{\lambda_2}^+(x) \setminus B_{\lambda}^+(x)} (u_{x,\lambda} + \tau u)^{\frac{4}{n-2}} |(u_{x,\lambda} - u)^+|^2 \\ &\leq C(n) \left(\int_{B_{\lambda_2}(x)} u^{\frac{2n}{n-2}} dy \right)^{2/n} \left(\int_{B_{\lambda_2}^+(x) \setminus B_{\lambda}^+(x)} |(u_{x,\lambda} - u)^+|^{\frac{2n}{n-2}} \right)^{(n-2)/n}, \end{aligned} \quad (31)$$

where $C(n) = n(n-2)2^{\frac{4}{n-2}}$. Since $(u_{x,\lambda} - u)^+ \in H_0^1(B_{\lambda_2}^+(x) \setminus B_{\lambda}^+(x))$, by Sobolev inequality we have

$$\int_{B_{\lambda_2}^+(x) \setminus B_{\lambda}^+(x)} |\nabla(u_{x,\lambda} - u)^+|^2 \geq \frac{1}{S(n)} \left(\int_{B_{\lambda_2}^+(x) \setminus B_{\lambda}^+(x)} |(u_{x,\lambda} - u)^+|^{\frac{2n}{n-2}} \right)^{(n-2)/n},$$

where $S(n) > 0$ depends only dimension n . Choosing λ_2 small to ensure

$$C(n) \left(\int_{B_{\lambda_2}(x)} u^{\frac{2n}{n-2}} dy \right)^{2/n} \leq \frac{1}{2S(n)}, \quad (32)$$

we obtain

$$\frac{1}{2S(n)} \left(\int_{B_{\lambda_2}^+(x) \setminus B_{\lambda}^+(x)} |(u_{x,\lambda} - u)^+|^{\frac{2n}{n-2}} \right)^{(n-2)/n} \leq 0,$$

which implies $u_{x,\lambda} \leq u$ in $B_{\lambda_2}^+(x) \setminus B_{\lambda}^+(x)$ because $(u_{x,\lambda} - u)^+$ is continuous in $B_{\lambda_2}^+(x) \setminus B_{\lambda}^+(x)$. Let $\lambda_1 = \lambda_3$ and we complete the proof.

Proof of (c). By the previous step, we see that $\bar{\lambda}_x$ is well defined. If $\bar{\lambda}_x < |x|$, we have $u - u_{x,\bar{\lambda}_x} \geq 0$ in $B_{3/4}^+ \setminus B_{\bar{\lambda}_x}(x)$. By item (a) and strong maximum principle $u - u_{x,\bar{\lambda}_x} > 0$ in $B_{3/4}^+ \setminus \bar{B}_{\bar{\lambda}_x}(x)$. It follows from Lemma 2.2 that for every $r > \bar{\lambda}_x$ there exists $c_r > 0$ such that

$$(u - u_{x,\bar{\lambda}_x})(y) \geq c_r y_n \quad \forall |y - x| \geq r, y \in B_{3/4}. \quad (33)$$

For $y \in B_{3/4}$ with $|y - x| \geq r$ and $\bar{\lambda}_x < \lambda < r \leq \frac{|x| + \bar{\lambda}_x}{2}$, making use of Lemma 2.1 we have

$$\begin{aligned} |u_{x,\bar{\lambda}_x}(y) - u_{x,\lambda}(y)| &\leq |y - x|^{2-n} |\bar{\lambda}_x^{n-2} - \lambda^{n-2}| u(x + \frac{\bar{\lambda}_x^2(y-x)}{|y-x|^2}) \\ &\quad + (\frac{\lambda}{|y-x|})^{n-2} |u(x + \frac{\bar{\lambda}_x^2(y-x)}{|y-x|^2}) - u(x + \frac{\lambda^2(y-x)}{|y-x|^2})| \\ &\leq C(\lambda - \bar{\lambda}_x)^\alpha y_n, \end{aligned} \quad (34)$$

where $\alpha \in (0, 1)$ and $C \geq 1$ depend only on n and $\|u\|_{L^\infty(B_{(|x|+\bar{\lambda}_x)/2}(x))}$, and we have used

$$u(x + \frac{\lambda^2(y-x)}{|y-x|^2}) \leq C \frac{\lambda^2 y_n}{|y-x|^2}$$

and

$$\begin{aligned} &|u(x + \frac{\bar{\lambda}_x^2(y-x)}{|y-x|^2}) - u(x + \frac{\lambda^2(y-x)}{|y-x|^2})| \\ &= \frac{\bar{\lambda}_x^2 y_n}{|y-x|^2} \left| \frac{u(x + \frac{\bar{\lambda}_x^2(y-x)}{|y-x|^2})}{\frac{\bar{\lambda}_x^2 y_n}{|y-x|^2}} - \frac{u(x + \frac{\lambda^2(y-x)}{|y-x|^2})}{\frac{\bar{\lambda}_x^2 y_n}{|y-x|^2}} \right| \\ &\leq C y_n \left| \frac{u(x + \frac{\bar{\lambda}_x^2(y-x)}{|y-x|^2})}{\frac{\bar{\lambda}_x^2 y_n}{|y-x|^2}} - \frac{u(x + \frac{\lambda^2(y-x)}{|y-x|^2})}{\frac{\lambda^2 y_n}{|y-x|^2}} \right| + C y_n \frac{\lambda^2 - \bar{\lambda}_x^2}{\bar{\lambda}_x^2} \\ &\leq C(\lambda - \bar{\lambda}_x)^\alpha y_n. \end{aligned}$$

Let $\delta > 0$ satisfy

$$C\delta^\alpha < \frac{1}{2}c_r, \quad \delta < \frac{|x| - \bar{\lambda}_x}{2}. \quad (35)$$

Then by (33) we have for all $\bar{\lambda}_x \leq \lambda \leq \bar{\lambda}_x + \delta$

$$(u - u_{x,\lambda})(y) \geq \frac{1}{2}c_r y_n \quad \forall |y - x| \geq r, y \in B_{3/4}. \quad (36)$$

This implies that $u - u_{x,\lambda} \geq 0$ on $\partial(B_r^+(x) \setminus B_\lambda^+(x))$. Using *narrow domain technique* as before, we immediately obtain

$$u \geq u_{x,\lambda} \quad \text{in } B_r^+(x) \setminus B_\lambda^+(x),$$

whenever r is chosen such that

$$n(n-2)2^{\frac{4}{n-2}} \left(\int_{B_r^+(x) \setminus B_{\bar{\lambda}_x}^+(x)} u^{\frac{2n}{n-2}} \right)^{2/n} \leq \frac{1}{2S(n)}. \quad (37)$$

In conclusion,

$$u_{x,\lambda}(y) \leq u(y) \quad \forall \bar{\lambda}_x \leq \lambda \leq \bar{\lambda}_x + \delta, \quad y \in B_{3/4}^+ \setminus B_{\bar{\lambda}}^+(x).$$

This contradicts to the definition of $\bar{\lambda}_x$. Hence, $\bar{\lambda}_x = |x|$.

Therefore, we proved Proposition 5.1 when $\psi = 0$.

If $\psi \neq 0$ is concave, we note that for each $y \in \Gamma_1 \cap B_{\lambda}(x)$, where $x \in \Gamma_1$, $|x'| < \varepsilon$ and $\lambda < |x'|$, whenever $|y' + \frac{\lambda^2(y'-x')}{|y-x|^2}| \leq \frac{3}{4}$ then $y + \frac{\lambda^2(y-x)}{|y-x|^2} \in Q_{3/4}$. Hence, with a little modification of the above proof for case $\psi = 0$, we complete the proof of Proposition 5.1. \square

Proof of Proposition 1.1. We are going to show that for all $x \in \partial\mathbb{R}_+^n$, $x \neq 0$ there holds

$$u_{x,\lambda}(y) \leq u(y) \quad \forall 0 < \lambda < |x|, \quad |y-x| \geq \lambda. \quad (38)$$

The idea is the same as that of the proof of Proposition 5.1.

Step 1. We prove that (38) holds for all $0 < \lambda < \lambda_1$ with $\lambda_1 > 0$ small.

Corresponding to the step (a) of the proof of Proposition 5.1, by Lemma 2.4 and Lemma 2.1 we have for $0 < \lambda_3 < \lambda_2 < |x|$

$$u(y) \geq \frac{\lambda_2^n y_n}{|y-x|^n} \inf_{y \in \partial'' B_{\lambda_2}^+(x), y_n > 0} \frac{u(y)}{y_n} \quad (39)$$

and

$$\inf_{y \in \partial'' B_{\lambda_2}^+(x), y_n > 0} \frac{u(y)}{y_n} > 0.$$

It follows from Lemma 2.1 that

$$\begin{aligned} u_{x,\lambda}(y) &= \left(\frac{\lambda}{|y-x|} \right)^{n-2} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \\ &\leq C \left(\frac{\lambda}{|y-x|} \right)^{n-2} \frac{\lambda^2 y_n}{|y-x|^2} = C \frac{\lambda^n y_n}{|y-x|^n}. \end{aligned}$$

In view of (39), by setting $\lambda_3 \leq \lambda_2/C^{1/n}$ we showed that

$$u_{x,\lambda}(y) \leq u(y) \quad \text{for } 0 < \lambda < \lambda_3, \quad |y-x| \geq \lambda_2.$$

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As in the step (b) of the proof of Proposition 5.1, by narrow domain technique we can prove easily that

$$u_{x,\lambda}(y) \leq u(y) \quad \forall 0 < \lambda < \lambda_3, \lambda \leq |y - x| \leq \lambda_2,$$

where λ_2 is selected to ensure (32).

Step 2. Define

$$\bar{\lambda}_x = \sup\{0 < \lambda < |x| : u_{x,\mu}(y) \leq u(y) \quad \forall 0 < \mu < \lambda, y \in \mathbb{R}_+^n \setminus B_\mu^+(x)\}.$$

By the previous step, $\bar{\lambda}_x > 0$ is well defined. We shall prove $\bar{\lambda}_x = |x|$. If not, i.e., $\bar{\lambda}_x < |x|$, we want to show that there exists $0 < \delta < \frac{|x| - \bar{\lambda}_x}{2}$ such that (38) holds for all $0 < \lambda < \bar{\lambda}_x + \delta$. This obviously contradicts to the definition of $\bar{\lambda}_x$.

By the definition of $\bar{\lambda}_x$, we have $u - u_{x,\bar{\lambda}_x} \geq 0$ in $\mathbb{R}_+^n \setminus B_{\bar{\lambda}_x}(x)$ and thus

$$-\Delta(u - u_{x,\bar{\lambda}_x}) \geq 0.$$

Since 0 is a non-removable singularity of u , we have $\limsup_{y \rightarrow 0} (u - u_{x,\bar{\lambda}_x})(y) = \infty$. By strong maximum principle, we have

$$u - u_{x,\bar{\lambda}_x} > 0 \quad \text{in } \mathbb{R}_+^n \setminus \bar{B}_{\bar{\lambda}_x}(x).$$

By Lemma 2.4 and Lemma 2.1,

$$u(y) \geq \frac{r^n y_n}{|y - x|^n} \inf_{y \in \partial'' B_r^+(x), y_n > 0} \frac{u(y)}{y_n} \quad (40)$$

and

$$\inf_{y \in \partial'' B_r^+(x), y_n > 0} \frac{u(y)}{y_n} > 0$$

for every $r > \bar{\lambda}_x$. r will be fixed to ensure (37) when using the *narrow domain technique*. Choosing $0 < \delta < \frac{|x| - \bar{\lambda}_x}{2}$ sufficiently small ensures that

$$|u_{x,\lambda}(y) - u_{x,\bar{\lambda}_x}(y)| \leq \frac{1}{2} \frac{r^n y_n}{|y - x|^n} \inf_{y \in \partial'' B_r^+(x), y_n > 0} \frac{u(y)}{y_n} \quad \forall y \in \mathbb{R}_+^n, |y| \geq r, \bar{\lambda}_x \leq \lambda \leq \bar{\lambda}_x + \delta. \quad (41)$$

Indeed, notice that $x + \frac{\lambda^2(y-x)}{|y-x|^2} \in B_{\frac{|x|+\bar{\lambda}_x}{2}}^+(x)$. By Lemma 2.1 and computing as in deriving (34) we have

$$\begin{aligned} |u_{x,\bar{\lambda}_x}(y) - u_{x,\lambda}(y)| &\leq |y - x|^{2-n} |\bar{\lambda}_x^{n-2} - \lambda^{n-2}| u\left(x + \frac{\bar{\lambda}_x^2(y-x)}{|y-x|^2}\right) \\ &\quad + \left(\frac{\lambda}{|y-x|}\right)^{n-2} \left|u\left(x + \frac{\bar{\lambda}_x^2(y-x)}{|y-x|^2}\right) - u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right)\right| \\ &\leq C \delta^\alpha \frac{y_n}{|y-x|^n}, \end{aligned}$$

where $\alpha \in (0, 1)$ and $C \geq 1$ depend only on n and $\|u\|_{L^\infty(B_{(|x|+\bar{\lambda}_x)/2})}$. Hence, (41) holds by setting

$$C\delta^\alpha \leq \frac{r^n}{2} \inf_{y \in \partial'' B_r^+(x), y_n > 0} \frac{u(y)}{y_n}.$$

Hence,

$$u_{x,\lambda}(y) \leq u(y) \quad \forall y \in \mathbb{R}_+^n, |y| \geq r, \bar{\lambda}_x \leq \lambda \leq \bar{\lambda}_x + \delta < r.$$

By *narrow domain technique*, the above inequality holds for all $y \in \mathbb{R}_+^n$ with $|y| \geq \lambda$. Therefore, step 2 is finished.

Let $e = (e', 0) \in \mathbb{R}^n$ be an arbitrary unit vector, $a > 0$ constant, and $y \in \mathbb{R}_+^n$ satisfying $ye - a < 0$, (38) holds for $x = Re$ and $\lambda = R - a$:

$$u(y) \geq \left(\frac{R-a}{|y-Re|}\right)^{n-2} u\left(x + \frac{(R-a)^2(y-Re)}{|y-Re|^2}\right).$$

Sending $R \rightarrow \infty$, we have

$$u(y) \geq u(y - 2(y \cdot e - a)e) = u(y' - 2(y' \cdot e' - a)e', y_n).$$

Proposition 1.1 follows immediately. □

Proposition 5.2. *Let $u \in C^2(\bar{Q}_1 \setminus \{0\})$ be a nonnegative solution of (4). Then there exists a constant $\bar{\delta} > 0$ depends on the C^2 norm of ψ such that for $x \in Q_{1/2}$ with $\frac{d(x)}{|x|} < \gamma < 1$ and $x_n > \frac{1}{\bar{\delta}} \max_{|y'|=|x'|} |\psi(y')|$ we have*

$$u(x', x_n) = \bar{u}(|x'|, x_n)(1 + O(|x|)), \quad (42)$$

where $\bar{u}(x', x_n) = \int_{\mathbb{S}^{n-2}} u(|x'|\theta, x_n) d\theta$ and $O(|x|) \leq C(\gamma)|x|$ as $x \rightarrow 0$.

Proof. Suppose first that ψ is concave. For $r > 0$, let $x_\alpha = (x'_\alpha, x_n)$ and $x_\beta = (x'_\beta, x_n)$ be two points in Q_ε with $x_n > \max_{|y'|=r} \psi(y')$ such that

$$u(x_\alpha) = \max_{|x'|=r} u(x', x_n) \quad \text{and} \quad u(x_\beta) = \min_{|x'|=r} u(x', x_n),$$

where ε is the one in Proposition 5.1. Let

$$x'_\gamma = x'_\alpha + \frac{\varepsilon(x'_\alpha - x'_\beta)}{4|x'_\alpha - x'_\beta|} \quad \text{and} \quad x_\gamma = (x'_\gamma, \psi(x'_\gamma)).$$

We want to find $\tilde{x}_\beta = (x'_\beta, t)$ and $\lambda > 0$ such that

$$x_\gamma + \frac{\lambda^2(\tilde{x}_\beta - x_\gamma)}{|\tilde{x}_\beta - x_\gamma|^2} = x_\alpha.$$

It follows that

$$t = \frac{4}{\varepsilon}(x_n - \psi(x'_\gamma))|x'_\alpha - x'_\beta| + x_n$$

and

$$\lambda^2 = \frac{\varepsilon(|x'_\beta - x'_\gamma|^2 + \varepsilon^{-2}(x_n - \psi(x'_\gamma))^2(4|x'_\alpha - x'_\beta| + \varepsilon)^2)}{4|x'_\alpha - x'_\beta| + \varepsilon}.$$

It is easy to check that there exist positive constants \bar{r} and $\bar{C}(\varepsilon)$, depending only on ε and $\psi(x'_\gamma)$, such that that if $|x_n| \leq \bar{C}(\varepsilon)\sqrt{\bar{r}}$ and $r < \bar{r}$ then $\lambda^2 < |x_\gamma|^2$. By Proposition 5.1, we have

$$\left(\frac{\lambda}{|\tilde{x}_\beta - x_\gamma|}\right)^{n-2}u(x_\alpha) = u_{x_\gamma, \lambda}(\tilde{x}_\beta) \leq u(\tilde{x}_\beta).$$

Let $s = \sqrt{r^2 + x_n^2}$ and $u_s(x) = s^{\frac{n-2\sigma}{2}}u(sx)$. Then

$$-\Delta u_s = n(n-2)u_s^{\frac{n+2}{n-2}} =: V(x)u_s \quad \text{in } \tilde{Q}_2 \setminus \tilde{Q}_{1/2}, \quad u_s = 0 \quad \text{on } \tilde{\Gamma}_2 \setminus \tilde{\Gamma}_{1/2}, \quad (43)$$

where $V(x) := n(n-2)u_s^{\frac{4}{n-2}}$ and \tilde{Q}_R and $\tilde{\Gamma}_R$ are the scalings of Q_{sR} and $\Gamma(sR)$. Let $\Omega_\gamma = \{x \in \tilde{Q}_{3/2} \setminus \tilde{Q}_{3/4} : \frac{d_{sx}}{|s|x|} < \gamma\}$. By elliptic estimates to the boundary and Lemma 2.1, we have

$$|\nabla u_s(\frac{1}{s}(x_\beta + \theta\tilde{x}_\beta))| \leq \frac{C \sup_{\Omega_\gamma} u_s}{\text{dist}(\frac{1}{s}(x_\beta + \theta\tilde{x}_\beta), \tilde{\Gamma}_2 \setminus \tilde{\Gamma}_{1/2})} \leq \frac{Cu_s(\frac{1}{s}\tilde{x}_\beta)}{\text{dist}(\frac{1}{s}\tilde{x}_\beta, \tilde{\Gamma}_2 \setminus \tilde{\Gamma}_{1/2})}$$

for every $\theta \in (0, 1)$, where C depends only on n, ψ and the constant $C(\frac{\gamma+1}{2})$ in Proposition 3.2.

Since $|\tilde{x}_\beta - x_\beta| = \frac{4(x_n - \psi(x'_\gamma))|x'_\alpha - x'_\beta|}{\varepsilon}$, by mean value theorem we have

$$\left|\frac{u(x_\beta)}{u(\tilde{x}_\beta)} - 1\right| \leq Cr.$$

Hence,

$$\begin{aligned} u(x_\alpha) &\leq u(x_\beta)(1 + Cr)\left(\frac{|\tilde{x}_\beta - x_\gamma|}{\lambda}\right)^{n-2} \\ &= u(x_\beta)(1 + Cr)\left(\frac{4|x'_\alpha - x'_\beta| + \varepsilon}{\varepsilon}\right)^{\frac{n-2}{2}} \\ &= u(x_\beta)(1 + O(r)). \end{aligned}$$

Therefore, the proposition is proved if ψ is concave.

If ψ is not concave, let $B_\rho(\rho e_n)$ be an inner tangential ball of Q_1 contacting Q_1 at 0, where $\rho > 0$ for Γ_1 is of C^2 . Let

$$\tilde{\phi}(y) = \rho e_n + \frac{\rho^2(y - \rho e_n)}{|y - \rho e_n|^2}, \quad v(y) = \left(\frac{\rho}{|y - \rho e_n|}\right)^{n-2}u(\rho e_n + \frac{\rho^2(y - \rho e_n)}{|y - \rho e_n|^2}),$$

$D = \phi^{-1}Q_1$ and $\Lambda = \phi^{-1}\Gamma_1$. Then $0 \in \Lambda$, D is concave at 0 and

$$\Delta v = n(n-2)v^{\frac{n+2}{n-2}} \quad \text{in } D, \quad v = 0 \quad \text{on } \Lambda.$$

Note that $|x| \leq C(\rho)|y|$ for $|y| \leq \frac{\rho}{100}$, and

$$\frac{\rho}{|y - \rho e_n|} = \rho^{-1} \left| \frac{\rho^2(y - \rho e_n)}{|y - \rho e_n|} + \rho e_n - \rho e_n \right| = |\rho^{-1}x - e_n| = 1 + O(|x|),$$

where $x = \frac{\rho^2(y - \rho e_n)}{|y - \rho e_n|} + \rho e_n$. By what we proved for concave ψ , the proposition follows immediately. \square

Proposition 5.3. *Suppose that u is a solution of (4) and $u(x) \leq Cd(x)|x|^{-\frac{n}{2}}$. Then there exists a constant $\bar{\delta} > 0$ depends on the C^2 norm of ψ such that*

$$u(x', x_n) = \bar{u}(|x'|, x_n)(1 + O(|x|)) \quad \text{as } x \rightarrow 0 \text{ with } x_n > \frac{1}{\bar{\delta}} \max_{|y'|=|x'|} |\psi(y')|. \quad (44)$$

Proof. By the proof of Proposition 5.2, we only consider concave ψ and $x_n \geq \frac{1}{C}\sqrt{|x'|}$ for some $\tilde{C} > 0$.

For $r > 0$ and $x_n \geq \frac{1}{C}\sqrt{r}$, let $x_\alpha = (x'_\alpha, x_n)$ and $x_\beta = (x'_\beta, x_n)$ be two points such that

$$u(x_\alpha) = \max_{|x'|=r} u(x', x_n) \quad \text{and} \quad u(x_\beta) = \min_{|x'|=r} u(x', x_n).$$

Let u_s satisfy (43) with $s = \sqrt{r^2 + |x_n|^2}$. By mean value theorem, Harnack inequality and interior estimates, we have

$$|u_s(\frac{1}{s}x_\alpha) - u_s(\frac{1}{s}x_\beta)| \leq Cu_s(\frac{1}{s}x_\beta)\frac{1}{s}|x_\alpha - x_\beta| \leq Cu_s(\frac{1}{s}x_\beta)s,$$

where we used $x_n \geq \frac{1}{C}\sqrt{r}$. It follows that

$$u(x_\alpha) = u(x_\beta)(1 + O(s)).$$

We complete the proof. \square

Proof of Theorem 1.2. The first part of the theorem follows from Proposition 3.2 and Proposition 5.2. The second part follows from Proposition 4.2, Proposition 4.3 and Proposition 5.3. \square

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